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Anomalous gauge theories revisited

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Abstract: A possible formulation of chiral gauge theories with an anomalous fermion content is re-examined in light of the lattice framework based on the Ginsparg-Wilson relation. It is shown that the fermion sector of a wide class of anomalous non-abelian theories cannot consistently be formulated within this lattice framework. In particular, in 4 dimension, *all* anomalous non-abelian theories are included in this class. Anomalous abelian chiral gauge theories cannot be formulated with compact U(1) link variables, while a non-compact formulation is possible at least for the vacuum sector in the space of lattice gauge fields. Our conclusion is not applied to effective low-energy theories with an anomalous fermion content which are obtained from an underlying anomaly-free theory by sending the mass of some of fermions to infinity. For theories with an anomalous fermion content in which the anomaly is cancelled by the Green-Schwarz mechanism, a possibility of a consistent lattice formulation is not clear.

Keywords: Renormalization Regularization and Renormalons, Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories.

Contents

1. Introduction

There had emerged a general belief that anomaly-free chiral gauge theories can non-perturbatively be formulated, after a work by Lüscher [1] which successfully formulated anomaly-free abelian chiral gauge theories in a gauge invariant manner on a lattice. Although there remain several challenging problems to be solved for a non-abelian extension of this work, the problems are well-posed and the general framework [1, 2], that is based on the so-called Ginsparg-Wilson relation [3], is ingenious. In fact, there already appeared several theoretical applications of this framework for general chiral gauge theories [4, 5, 6]. References [7]–[14] are related works for this development.

The target of the lattice framework [1, 2] is anomaly-free chiral gauge theories. On the other hand, in the context of continuum theory, it has sometimes been claimed that a consistent quantization of chiral gauge theories is possible even when the gauge anomaly is not cancelled [15, 16, 17]. The purpose of the present paper is to clarify a situation concerning these *anomalous gauge theories*, if one considers its quantization in the non-perturbative lattice framework of references [1, 2]. Actually, most of expressions below (except those in the second half of section 5) has already been known in the literature. However, since they are scattered in various places where a quantization of anomalous gauge theories is not the main subject, it seems useful to gather them together and summarize the situation in a compact form.

In short, our general conclusion is the following: Generally, chiral gauge theories with an anomalous fermion content cannot consistently be formulated along a line of the framework of references [1, 2]. This conclusion is not related to (at least to our present understanding) a loss of unitarity and/or renormalizability that is naively expected for gauge theories with anomalies. Our conclusion is not even related to, at least apparently, a loss

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of gauge invariance. Our point is that the partition function of the fermion sector or more generally expectation values in the fermion sector cannot be defined as a single-valued smooth function of the gauge fields, if one applies the framework [1, 2] to theories with an anomalous fermion content. This kind of situation does not happen in the continuum theory in which a formal integration over fermion variables always defines a single-valued smooth function of the gauge fields. This difference from the continuum theory arises because in the framework of references [1, 2], one has to be involved with a U(1) bundle associated to the fermion integration measure. A variation of the partition function is related to the U(1) connection of this bundle, called the measure term. There exists a wide freedom to choose the measure term corresponding to a freedom to choose local counter terms. However, if the U(1) bundle turns out to be non-trivial, the partition function cannot be a single-valued smooth function of the gauge fields for any choice of the measure term. These points will be clarified in the next section.

More specifically, for non-abelian gauge theories, we can show the breaking of the smoothness occurs if $\pi_1(G) = 0$ and $\pi_{2n+1}(G) = \mathbb{Z}$, where G is the gauge group and $2n$ is the dimension of the euclidean space, and a fermion content exhibits an "irreducible" gauge anomaly.[1] These conditions are precisely identical to prerequisites for a geometrical characterization of the gauge anomaly studied in reference [18]; the phase of the chiral determinant possesses a non-trivial winding number along a gauge loop and this implies a non-contractible 2 sphere in the gauge orbit space $\mathcal{A}/\mathcal{G}$ ($\mathcal{A}$ denotes the space of gauge potentials and $\mathcal{G}$ the group of gauge transformations). These prerequisites are applied to all non-abelian anomalous gauge theories in 4 dimension so we conclude that *all* 4 dimensional non-abelian anomalous gauge theories cannot consistently be formulated with the present lattice framework (section 3).

For abelian theories, on the other hand, the situation depends on whether one uses compact link variables or non-compact variables for U(1) gauge fields. For the compact U(1) case, the partition function in anomalous theories cannot be a single-valued smooth function (section 4). On the other hand, for the non-compact U(1) case, one can arrange a suitable measure term to define a physically acceptable fermion partition function at least for the vacuum sector in the space of lattice gauge fields (section 5). This is not surprising because, with non-compact variables, the space of gauge fields (that is the base space of the U(1) bundle) is topologically trivial and, consequently, the U(1) bundle cannot be non-trivial.

We should emphasize that our conclusion is not applied to effective low-energy theories with an anomalous fermion content which are obtained by sending the mass of some of fermions infinity [19] from an underlying anomaly-free theory. This and related issues will be discussed in section 6.

The dimension of the euclidean space will be denoted as $d = 2n$. Lorentz indices, μ, ν, ... run from 0 to $d-1$. The hyper-cubic euclidean lattice (whose one-dimensional size is L) will be denoted by $\Gamma = \{x \in \mathbb{Z}^d \mid 0 \leq x_\mu < L\}$. The unit vector along the

[1] The irreducible here means that the anomaly cannot be expressed as a polynomial of strictly lower rank traces over the fundamental representation of the gauge group.

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direction μ will be denoted by $\hat{\mu}$. The lattice spacing is taken to be unity $a = 1$ unless stated otherwise. The symbol $\epsilon_{\mu_1\mu_2\cdots\mu_d}$ stands for the totally anti-symmetric tensor with $\epsilon_{01\cdots d-1} = 1$. We define the chiral matrix by $\gamma_5 = i^{-n}\gamma_0\gamma_1\cdots\gamma_{d-1}$ from hermitian Dirac matrices γ_μ. The forward and backward difference operators on a lattice are defined by $\partial_\mu f(x) = \{f(x+a\hat{\mu}) - f(x)\}/a$ and $\partial^*_\mu f(x) = \{f(x) - f(x-a\hat{\mu})\}/a$, respectively.

2. Brief review of the lattice framework for Weyl fermions [1, 2]

Our objective is the partition function, i.e. the Weyl determinant, of a (left-handed) Weyl fermion on a lattice:[2]

$$\det M[U] = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\, e^{-S_{\mathrm{F}}}\,, \qquad S_{\mathrm{F}} = \sum_{x\in\Gamma} \overline{\psi}(x) D\psi(x)\,, \tag{2.1}$$

where U denotes the lattice gauge field (link variables). A crucial assumption on the lattice Dirac operator D is the Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D\,, \tag{2.2}$$

and the simplest example of such a lattice operator is provided by the Neuberger overlap Dirac operator [20]. The overlap Dirac operator is free from the species doubling and possesses the γ_5-hermiticity ($D^\dagger = \gamma_5 D\gamma_5$), the gauge covariance and moreover the locality[3] provided that configurations of gauge fields are restricted by the admissibility [21]

$$\|\, 1 - R[U(p)]\, \| < \epsilon\,, \qquad \text{for all plaquettes } p\,, \tag{2.3}$$

where $U(p)$ is the product of link variables around the plaquette p; R stands for a (generally reducible) unitary representation of the gauge group to which the Weyl fermion belongs (ϵ is a certain fixed constant). This condition divides otherwise arc-wise connected space of lattice gauge fields into disconnected topological sectors.

The Ginsparg-Wilson relation allows one to introduce the modified chiral matrix $\hat{\gamma}_5 = \gamma_5(1-D)$ which fulfills

$$(\hat{\gamma}_5)^\dagger = \hat{\gamma}_5\,, \qquad (\hat{\gamma}_5)^2 = 1\,, \qquad D\hat{\gamma}_5 = -\gamma_5 D\,. \tag{2.4}$$

Therefore, especially due to the last relation, the chirality of the Weyl fermion can consistently be defined in the lattice action S_{F} by imposing the constraints

$$\hat{P}_-\psi = \psi, \qquad \overline{\psi} = \overline{\psi}P_+\,, \tag{2.5}$$

where we have introduced projection operators by $\hat{P}_\pm = (1\pm\hat{\gamma}_5)/2$ and $P_\pm = (1\pm\gamma_5)/2$. Note that the first constraint depends on lattice gauge fields through the Dirac operator D. The classical lattice action S_{F} for the Weyl fermion is then gauge invariant and local due to properties of D listed above.

[2]Our description will be rather brief. For more details, readers are referred to references [1, 2].

[3]For a precise definition of the locality assumed here, see reference [1].

We have to next define the fermion integration measure $\mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]$ in equations (2.1). For this, we introduce an orthonormal complete set of vectors in the constrained space (2.5):

$$\hat{P}_{-} v_j = v_j\,, \qquad (v_k, v_j) = \delta_{kj}\,, \tag{2.6}$$

and expand the fermion field ψ in terms of this basis as $\psi(x) = \sum_j v_j(x)c_j$. The integration measure is then defined by $\mathrm{D}[\psi] = \prod_j \mathrm{d}c_j$.[4] In terms of these bases, the Weyl determinant (2.1) is given by the determinant of the matrix $M_{kj} = \sum_{x\in\Gamma} \overline{v}_k(x)Dv_j(x)$.

The above construction, although seemingly simple, turns out to be rather involved due to following facts. First, the constraint (2.6) does not specify the basis vectors uniquely. A different choice of basis leads to a difference in the phase of the fermion integration measure and this phase ambiguity may influence on physical contents of the system because the phase may depend on gauge field configurations. One has to fix this phase ambiguity so that basic physical requirements (smoothness, locality etc.) are fulfilled. Secondly, a connected component of the space of admissible gauge fields, denoted by $\mathfrak{U}$, may have a non-trivial topological structure. Hence it is not obvious if there exists a basis over $\mathfrak{U}$ with which above physical requirements are fulfilled. These problems can be formulated as follows [1, 2].

We cover $\mathfrak{U}$ by local coordinate patches X_A, where A stands for a label of patches. Suppose that a certain basis v_j^A has been chosen within each of patch. Generally, however, on the intersection $X_A \cap X_B$ of two patches, basis vectors v_j^A and v_j^B are different; in general, these two are related by a unitary transformation

$$v_j^B(x) = \sum_k v_k^A(x)\tau(A \to B)_{kj}\,. \tag{2.7}$$

By definition, the transition function $\tau(A \to B)$ satisfies the cocycle condition and thus it defines a unitary principal bundle over $\mathfrak{U}$. Corresponding to the relation (2.7), the fermion integration measures defined in X_A and in X_B are related as

$$\mathrm{D}[\psi]_B = \mathrm{D}[\psi]_A g_{AB}\,, \qquad g_{AB} = \det\tau(A \to B) \in \mathrm{U}(1)\,. \tag{2.8}$$

This phase factor g_{AB} thus defines a U(1) bundle associated to the fermion integration measure. For the present formulation of the Weyl fermion to be meaningful, the partition function (2.1) (or more generally expectation values in the fermion sector) must be a single-valued smooth function over $\mathfrak{U}$. To realize this, we re-define basis vectors in each patch[5] such that $g_{AB} = 1$ on all intersections. However, *this is possible if and only if the* U(1) *bundle is trivial.*

To find a characterization of this U(1) bundle, we consider a variation of gauge field

$$\delta_\eta U(x,\mu) = \eta_\mu(x)U(x,\mu)\,, \qquad \eta_\mu(x) = \eta_\mu^a(x)T^a\,, \tag{2.9}$$

[4]Similarly, the anti-fermion $\overline{\psi}$ is expanded by a basis such that $\overline{v}_k P_+ = \overline{v}_k$ as $\overline{\psi}(x) = \sum_k \overline{c}_k \overline{v}_k(x)$ and the integration measure is defined by $\mathrm{D}[\overline{\psi}] = \prod_k \mathrm{d}\overline{c}_k$.

[5]Under a change of basis, the phase factor transforms as $g_{AB} \to h_A g_{AB} h_B^{-1}$, where h_A is the determinant of the transformation matrix between two bases in X_A.

and define the "measure term" within a patch, say X_A, by

$$\mathfrak{L}_\eta^A = i \sum_j (v_j^A, \delta_\eta v_j^A)\,. \tag{2.10}$$

From the definition of the Weyl determinant, we have

$$\delta_\eta \ln\det M = \mathrm{Tr}\left\{\delta_\eta D \hat{P}_- D^{-1} P_+\right\} - i\mathfrak{L}_\eta^A\,, \tag{2.11}$$

and this shows that for the Weyl determinant to be a single-valued smooth function over $\mathfrak{U}$, the measure term $\mathfrak{L}_\eta$ must be smoothly defined over $\mathfrak{U}$. On the intersection, $X_A \cap X_B$, we see from equations (2.7) that the measure terms in X_A and in X_B are related as

$$\mathfrak{L}_\eta^B = \mathfrak{L}_\eta^A + i g_{AB}^{-1} \delta_\eta g_{AB}\,. \tag{2.12}$$

This shows that the measure term is the U(1) connection associated to the U(1) bundle. Applying the identity $\delta_\eta\delta_\zeta - \delta_\zeta\delta_\eta + \delta_{[\eta,\zeta]} = 0$ to the definition of the measure term (2.10), we find[6]

$$\delta_\eta \mathfrak{L}_\zeta^A - \delta_\zeta \mathfrak{L}_\eta^A + \mathfrak{L}_{[\eta,\zeta]}^A = i\,\mathrm{Tr}\left\{\hat{P}_-\left[\delta_\eta \hat{P}_-\,, \delta_\zeta \hat{P}_-\right]\right\}. \tag{2.13}$$

The right hand side is nothing but the curvature of the U(1) bundle. Thus we can define the first Chern number of the U(1) bundle:

$$\mathcal{I} = \frac{1}{2\pi}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}s\, i\,\mathrm{Tr}\left\{\hat{P}_-\left[\delta_\eta \hat{P}_-\,, \delta_\zeta \hat{P}_-\right]\right\}, \tag{2.14}$$

which is an *integer*. $\mathcal{M}$ in this expression stands for a *closed* 2 dimensional surface in $\mathfrak{U}$ and t and s are local coordinates of $\mathcal{M}$. Namely, we defined 2 parameter family of gauge fields $U_{t,s}$ in $\mathfrak{U}$, and accordingly introduced the projection operators by $P_{t,s} = \hat{P}_-|_{U=U_{t,s}}$. The variations are given by $\eta_\mu(x) = \partial_t U_{t,s}(x,\mu) U_{t,s}(x,\mu)^{-1}$ and $\zeta_\mu(x) = \partial_s U_{t,s}(x,\mu) U_{t,s}(x,\mu)^{-1}$. The important point for discussions in this paper is that if the above integer $\mathcal{I}$ does not vanish, then the U(1) bundle is non-trivial on $\mathcal{M}$ and there exists *no* possible choice of basis vectors such that the partition function (or expectation values in the fermion sector) is a single-valued smooth function over $\mathcal{M}$. Namely, the Weyl fermion cannot consistently be formulated. In this case, $\mathcal{M}$ is a non-contractible 2 surface in $\mathfrak{U}$ which gives rise to a topological obstruction in defining a smooth fermion measure.

In the above argument, we started with basis vectors defined patch by patch. This way of argument, however, is not convenient in constructing the fermion measure which is consistent with the locality. The measure term $\mathfrak{L}_\eta$, being linear in the variation η, can be expressed as

$$\mathfrak{L}_\eta = \sum_{x\in\Gamma} \eta_\mu^a(x) j_\mu^a(x)\,, \tag{2.15}$$

where $j_\mu^a(x)$ is referred to as the measure current. It can be shown [1] that the expected locality of the system is guaranteed if the measure current is a local function of lattice gauge fields. To ensure the correct physical contents of the formulation, therefore, it is

[6] Here we have assumed that the variations η and ζ are independent of the gauge field.

convenient to start with a certain current which is a local function of gauge fields. Then according to the reconstruction theorem [1, 2], if the local current satisfies the following conditions, one can re-construct basis vectors which lead to a smooth fermion measure that is consistent with the locality. (1) The current is a single-valued and smooth function of the gauge fields contained in $\mathfrak{U}$. (2) The measure term (2.15) defined from the current satisfies the global integrability

$$\exp\left\{i\int_0^1 \mathrm{d}t\,\mathfrak{L}_\eta\right\} = \det\{1 - P_0 + P_0 Q_1\}\,, \tag{2.16}$$

along *any* closed loop in $\mathfrak{U}$. In this expression, the loop U_t ($U_1 = U_0$) is parametrized by $t \in [0,1]$ and the variation is given by $\eta_\mu(x) = \partial_t U_t(x,\mu) U_t(x,\mu)^{-1}$. Projection operators along the loop are defined by $P_t = \hat{P}_-|_{U=U_t}$. The operator Q_t is defined by the differential equation $\partial_t Q_t = [\partial_t P_t, P_t] Q_t$ and $Q_0 = 1$. Thus a task to construct the fermion measure is reduced to find an appropriate local current $j^a_\mu(x)$ which satisfies the above two conditions.[7] For a contractible loop in $\mathfrak{U}$, it can be shown that the global integrability is equivalent to the local integrability

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + \mathfrak{L}_{[\eta,\zeta]} = i\,\mathrm{Tr}\{\hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-]\}\,, \tag{2.17}$$

that is nothing but equations (2.13) without patch labels (because here we started with a measure term globally defined over $\mathfrak{U}$).

3. Non-abelian anomalous theories

In this section, we assume that the gauge group G is semi-simple. For these non-abelian gauge theories, we arrange following 2 parameter family of gauge field configurations for which we can evaluate $\mathcal{I}$ (2.14) (in this case, $\mathcal{M} = S^2$). First, we consider a 1 parameter family of gauge transformations defined on S^{2n} and take a lattice transcription of those gauge transformations:

$$g_t(x) \in G, \qquad x \in \Gamma, \qquad 0 \le t \le 1, \qquad\qquad g_0(x) = g_1(x) = 1\,. \tag{3.1}$$

From these lattice gauge transformations, we define a 2 parameter family of gauge field configurations:[8]

$$U_{t,s}(x,\mu) = \left[g_t^{-1}(x) g_t(x+\hat{\mu})\right]^s\,, \qquad 0 \le t \le 1\,, \qquad 0 \le s \le 1\,. \tag{3.2}$$

[7] For a gauge invariant formulation for anomaly-free chiral gauge theories, an additional condition, the anomaly cancellation on the lattice [1, 2], has also to be fulfilled.

[8] Group elements, except those in a measure-zero set in G, are uniquely parametrized as $g = \exp(\theta^a T^a)$ by using group generators T^a. Powers of an element g are then defined by $[g]^s = \exp(s\theta^a T^a)$. Powers in equation (3.2) can always be unambiguously defined at sufficiently small lattice spacings, because the inside of the powers behaves as $1 + O(a)$ for small lattice spacings. On the other hand, we assume that the 1 parameter family (3.1) has been chosen so that powers in equation (3.3) can be defined in the aforementioned way.

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In the continuum limit, these configurations reduce to linear interpolations between the trivial vacuum and its continuum gauge transformations defined by $g_t(x)$. These configurations satisfy the admissibility (2.3) if the lattice is fine enough, because the left hand side of equation (2.3) is bounded by an $O(a^2)$ quantity. This 2 parameter family thus spans a 2 disk D^2 in $\mathfrak{U}$. We also define another 2 parameter family of gauge field configurations:

$$U_{t,s}(x,\mu) = \left[g_{1-t}^{-1}(x)\right]^s \left[g_{1-t}(x+\hat{\mu})\right]^s , \qquad 0 \le t \le 1, \qquad 0 \le s \le 1 . \tag{3.3}$$

This 2 parameter family spans another 2 disk D^2 again in $\mathfrak{U}$, because these configurations are lattice gauge transformations of $U(x,\mu)=1$ and the admissibility is a gauge invariant condition trivially satisfied by $U(x,\mu) = 1$.[9] We then glue the above two disks (3.2) and (3.3) together along the edges $s=1$ and form a 2 sphere $\mathcal{M} = S^2$ in $\mathfrak{U}$.

The first Chern number $\mathcal{I}$ (2.14) is an integer. With the above setting, we can evaluate this integer in the classical continuum limit. The result is [6] (see also reference [22])

$$\begin{aligned}
\mathcal{I} = {} & \frac{(-1)^n i^{n+1} n!}{(2\pi)^{n+1}(2n+1)!} \int_{S^1\times S^{2n}} \mathrm{d}^{2n+1}x \, \times \\
& \qquad \times \epsilon_{\mu_1\mu_2\cdots\mu_{2n+1}} \operatorname{tr}\left\{R(g_t^{-1}\partial_{\mu_1} g_t) R(g_t^{-1}\partial_{\mu_2} g_t) \cdots R(g_t^{-1}\partial_{\mu_{2n+1}} g_t)\right\} \\
= {} & A_{n+1}(R) \frac{(-1)^n i^{n+1} n!}{(2\pi)^{n+1}(2n+1)!} \int_{S^1\times S^{2n}} \mathrm{d}^{2n+1}x \, \times \\
& \qquad \times \epsilon_{\mu_1\mu_2\cdots\mu_{2n+1}} \operatorname{tr}\left\{(g_t^{-1}\partial_{\mu_1} g_t)(g_t^{-1}\partial_{\mu_2} g_t) \cdots (g_t^{-1}\partial_{\mu_{2n+1}} g_t)\right\} ,
\end{aligned} \tag{3.4}$$

where we regarded t as an additional coordinate x_{2n}. In the second line, $A_{n+1}(R)$ is the leading anomaly coefficient in $2n$ dimensions, defined by[10]

$$\begin{aligned}
& \epsilon_{\mu_1\nu_1\cdots\mu_{n+1}\nu_{n+1}} \operatorname{tr}\{R(F_{\mu_1\nu_1})\cdots R(F_{\mu_{n+1}\nu_{n+1}})\} = \\
& \qquad = A_{n+1}(R)\epsilon_{\mu_1\nu_1\cdots\mu_{n+1}\nu_{n+1}} \operatorname{tr}\{F_{\mu_1\nu_1}\cdots F_{\mu_{n+1}\nu_{n+1}}\} + (\text{lower traces}) ,
\end{aligned} \tag{3.5}$$

where the trace in the right hand side is defined with respect to the fundamental representation of the gauge group. Note that, in 4 dimension, the non-abelian gauge anomaly is always “irreducible”, i.e. the gauge anomaly always implies $A_{n+1}(R) \neq 0$, because $\operatorname{tr} T^a = 0$ for non-abelian factors. To discuss in which circumstance the above integer $\mathcal{I}$ (3.4) does not vanish, we assume that $\pi_1(G) = 0$ and $\pi_{2n+1}(G) = \mathbb{Z}$. Then, the integer $\mathcal{I}$ (3.4) is $A_{n+1}(R)$ times the winding number of the gauge transformation $g_t(x)$ around the basic $(2n+1)$ sphere in G [18]. One can then pick up $g_t(x)$ on $S^1 \times S^{2n}$ for which $\mathcal{I} \neq 0$ if $A_{n+1}(R) \neq 0$.[11]

[9] These configurations, however, are a pure lattice construction and do not necessarily have a continuum limit. We notice that their continuum limit is not needed in the evaluation of equation (3.4) [6].

[10] For the equality of the first line of equation (3.4) to the second line, see, for example, reference [23].

[11] Take a topologically non-trivial mapping from S^{2n+1} to G which maps a certain point of S^{2n+1} to the identity. The mapping can be regarded as a mapping from a hypercube $B^{2n+1} = [0,1]^{2n+1}$ to G which maps all of the boundary ∂B^{2n+1} to the identity. We regard $B^{2n+1} = [0,1] \times B^{2n}$ and identity the first factor $[0,1]$ with the parameter t in equation (3.1). For each value of t, the mapping defined above maps all of the boundary ∂B^{2n} to the identity. So this provides a mapping $g_t(x)$ from $S^1 \times S^{2n}$ to G which gives a non-zero winding number.

In summary, if $\pi_1(G) = 0$, $\pi_{2n+1}(G) = \mathbb{Z}$ and the gauge anomaly is irreducible, i.e. $A_{n+1}(R) \neq 0$, then we can provide an explicit example of a 2 dimensional surface $\mathcal{M}$ in $\mathfrak{U}$ for which the first Chern number (2.14) is non-zero, $\mathcal{I} \neq 0$. We can thus infer that at least for such cases a consistent lattice formulation of non-abelian anomalous gauge theories along the line of framework of references [1, 2] is impossible, because the partition function (or expectation values in general) of the fermion sector cannot be a single-valued smooth function over $\mathcal{M}$. Although the above cases do not exhaust all possible non-abelian gauge theories with an anomalous fermion content, those are wide enough for one to expect a consistent formulation of non-abelian anomalous cases is probably impossible in general. In particular, in 4 dimension, non-abelian anomalies appear only when the gauge group G contains $\mathrm{SU}(n)$ ($n \geq 3$) as the factor (other groups have anomaly-free representations only). Since $\pi_1(\mathrm{SU}(n)) = 0$ and $\pi_5(\mathrm{SU}(n)) = \mathbb{Z}$ for $n \geq 3$, we conclude that *all* non-abelian anomalous theories in 4 dimension cannot consistently be formulated with the present lattice formulation.

4. Abelian theory with compact gauge variables

The obstruction for the smooth fermion measure discussed in the preceding section does not exist in abelian theories because the right hand side of equation (3.4) vanishes in this case. However, with a use of compact U(1) link variables, it has been pointed out that there exists a 2 torus $\mathcal{M} = T^2$ in $\mathfrak{U}$ for which the first Chern number (2.14) is non-zero for an anomalous fermion content. This implies that anomalous abelian chiral gauge theories cannot consistently be formulated within the present lattice framework if compact U(1) link variables are used.

The 2 torus in $\mathfrak{U}$ which is parametrized by coordinates t and s ($0 \leq t, s < 2\pi$) is defined by

$$U_{t,s}(x,\mu) = \exp\{i\delta_{\mu,0}\delta_{\tilde{x}_0,0}t + i\delta_{\mu,1}\delta_{\tilde{x}_1,0}s\}V_{[m]}(x,\mu)\,, \tag{4.1}$$

where $\tilde{x}_\mu = x_\mu \bmod L$, $0 \leq \tilde{x}_\mu < L$. The Wilson line $W_\mu(x) = \prod_{n=0}^{L-1} U(x+n\hat{\mu},\mu)$ at the origin of these configurations is given by $W_0(0) = e^{it}$ and $W_1(0) = e^{is}$. An important feature of these configurations is thus a winding along directions of the Wilson line in the space of lattice gauge fields. The factor $V_{[m]}(x,\mu)$ in the above expression carries a constant field strength $F_{\mu\nu} = (1/i)\ln P(x,\mu,\nu) = 2\pi m_{\mu\nu}/L^2$, where $m_{\mu\nu}$ are integers. For integers within a range $|\mathrm{e} m_{\mu\nu}| < \epsilon' L^2/2\pi$,[12] where e is the U(1) charge of the Weyl fermion, the configurations fulfill the admissibility (2.3). The explicit form of $V_{[m]}(x,\mu)$ is given by

$$V_{[m]}(x,\mu) = \exp\left\{-\frac{2\pi i}{L^2}\left[L\delta_{\tilde{x}_\mu,L-1}\sum_{\nu>\mu} m_{\mu\nu}\tilde{x}_\nu + \sum_{\nu<\mu} m_{\mu\nu}\tilde{x}_\nu\right]\right\}. \tag{4.2}$$

For the 2 parameter family of configurations (4.1), it can be shown that [24] (see also references [25, 22] for related study)

$$\mathcal{I} = \frac{(-1)^{n-1}\mathrm{e}^{n+1}}{2^{n-1}(n-1)!}\epsilon_{01\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}} m_{\mu_1\nu_1}\cdots m_{\mu_{n-1}\nu_{n-1}}\,, \tag{4.3}$$

[12] $\epsilon' = 2\arcsin(\epsilon/2)$.

when L, a one-dimensional size of the lattice, is large enough compared to the localization range of the Dirac operator. In particular, $\mathcal{I} = \mathrm{e}^2$ for $d = 2$. For the right-handed Weyl fermion, we have equation (4.3) with a minus sign. The Chern number (2.14) is therefore non-vanishing unless the gauge anomaly is cancelled among flavors of the Weyl fermion as $\sum_\alpha \mathrm{H}_\alpha \mathrm{e}_\alpha^{n+1} = 0$, where H_α and e_α are the chirality ($\mathrm{H}_\alpha = \pm 1$) and the U(1) charge of a flavor α, respectively.

In the present compact U(1) formulation, therefore, chiral gauge theories with an anomalous fermion content in general cannot consistently be formulated. This is a rather strong statement.

5. Abelian theory with non-compact gauge variables

The discussion in the preceding section does not exclude a possibility of a formulation based on *non-compact* U(1) gauge variables:

$$-\infty < A_\mu(x) < +\infty, \qquad x \in \Gamma, \qquad A_\mu(x + L\hat{\nu}) = A_\mu(x)\,, \tag{5.1}$$

because with these variables, a winding along directions of the Wilson line is topologically trivial (see below). The gauge transformation on the non-compact variables is defined by

$$A_\mu^\omega(x) = A_\mu(x) - \partial_\mu \omega(x)\,. \tag{5.2}$$

In this formulation, the gauge potentials $A_\mu(x)$ are regarded as primary dynamical variables and a gauge coupling of the lattice fermion is defined by substituting U(1) link variables in the lattice Dirac operator D by

$$R[U(x,\mu)] = e^{i\mathrm{e}A_\mu(x)}\,. \tag{5.3}$$

In the present non-compact formulation, the U(1) charge e is not necessarily an integer.

The admissibility (2.3) restricts possible configurations of the gauge potentials and divides the configuration space into disconnected sectors. Here, for simplicity, we consider the vacuum sector which is a connected component of the space of admissible configurations containing the trivial vacuum $A_\mu(x) = 0$. By defining the field strength by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\,, \tag{5.4}$$

the vacuum sector is characterized by the condition

$$|\mathrm{e}F_{\mu\nu}(x)| < \epsilon'\,. \tag{5.5}$$

The vacuum sector in this non-compact formulation is topologically trivial because when $A_\mu(x)$ satisfies equation (5.5), then $tA_\mu(x)$ with $0 \le t \le 1$ does too. Namely, any configuration in the vacuum sector can smoothly be deformed to the trivial one $A_\mu(x) = 0$ and, in particular, no non-trivial winding along directions of the Wilson line is possible. In what follows, we show that it is actually possible to set up a lattice formulation for anomalous U(1) chiral gauge theories with these non-compact gauge variables. For a related work for a lattice with an infinite extent, see reference [26].

We define the partition function of the whole system as follows

$$\mathcal{Z} = \int \mathrm{D}[A_\mu]\, e^{-S_{\mathrm{G}} - S_{\mathrm{counter}}} \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\, e^{-S_{\mathrm{F}}}\,, \tag{5.6}$$

where $\mathrm{D}[A_\mu] = \prod_x \mathrm{d}A_\mu(x)$ and, as the gauge action S_{G}, we adopt the modified plaquette action [1]

$$S_{\mathrm{G}} = \frac{1}{4g_0^2} \sum_{x\in\Gamma} \sum_{\mu,\nu} \mathcal{L}_{\mu\nu}\,, \tag{5.7}$$

where g_0 is the bare coupling and

$$\mathcal{L}_{\mu\nu} = \begin{cases} [F_{\mu\nu}(x)]^2 \left\{1 - \frac{[\mathrm{e}F_{\mu\nu}(x)]^2}{\epsilon'^2}\right\}^{-1} & \text{if } |\, \mathrm{e}F_{\mu\nu}(x)\,| < \epsilon' \\ \infty & \text{otherwise}\,. \end{cases} \tag{5.8}$$

The action S_{G} is designed to enforce the admissibility (5.5) dynamically.[13]

We next have to show that the fermion integration measure $\mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]$ can be defined smoothly on the space of admissible gauge fields, in a way being consistent with the locality. It is achieved by taking

$$\mathfrak{L}_\eta = i \int_0^1 \mathrm{d}t\ \mathrm{Tr}\left\{\hat{P}_- \left[\partial_t \hat{P}_-, \delta_\eta \hat{P}_-\right]\right\} + \delta_\eta \sum_{x\in\Gamma} A_\mu(x) \int_0^1 \mathrm{d}t\, k_\mu(x)\,, \tag{5.9}$$

as the measure term in equation (2.15). In this expression, the parameter t interpolates the gauge potential from 0 to $A_\mu(x)$, $A^t_\mu(x) = tA_\mu(x)$. It is understood that the gauge potentials $A_\mu(x)$ within the integration over t are $A^t_\mu(x)$. The current $k_\mu(x)$ is a smooth local function of the gauge potentials which will be explained below. It is then obvious that $\mathfrak{L}_\eta$ is a smooth function of admissible gauge fields. This is also consistent with the locality of the theory because the measure term $\mathfrak{L}_\eta$ consists of a lattice sum of local expressions of the gauge potentials due to the locality of the Dirac operator D. One can also verify that the measure term satisfies the local integrability condition (2.17). Since the space of admissible gauge fields is simply-connected in the present non-compact formulation, these guarantee that there exist basis vectors which lead a smooth fermion integration measure being consistent with the locality. This can easily be shown by constructing the partition function (2.1) explicitly which corresponds to the above choice of the measure term. That is given by

$$\ln\det M[A] = \int_0^1 \mathrm{d}t\ \mathrm{Tr}\left\{\partial_t D \hat{P}_- D^{-1} P_+\right\} - i \sum_{x\in\Gamma} A_\mu(x) \int_0^1 \mathrm{d}t\, k_\mu(x)\,. \tag{5.10}$$

From this expression, by using relations $\gamma_5 \delta D = -\delta D \hat{\gamma}_5 - D\delta\hat{\gamma}_5$ and $\hat{\gamma}_5 \delta P_- = -\delta\hat{P}_-\hat{\gamma}_5$, one finds equation (2.11) with the measure term (5.9). The above form of the measure term and the corresponding effective action have been known in the literature [1, 27]. We conclude therefore that the criticism to anomalous gauge theories in previous sections based on a smoothness of the fermion measure is not applied to the present case of non-compact U(1) variables.

[13] When the U(1) gauge field is coupled to several Weyl fermions with different U(1) charges, the U(1) charge with a maximal magnitude |e| is used in this expression.

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The partition function (5.10) is not invariant under the gauge transformation (5.2). We will see that the main part of the breaking is given by the "covariant gauge anomaly" defined by

$$\mathcal{A}(x) = -\frac{1}{2}\,\mathrm{tr}\{\gamma_5 \mathrm{e} D(x,x)\}\,. \tag{5.11}$$

A remarkable fact is that one can deduce the explicit form of $\mathcal{A}(x)$ in terms of the gauge potentials even on a finite lattice (as far as L is large enough) by using the local cohomology argument on a lattice [28]–[31]. Although the local cohomology argument was originally developed for the compact U(1) theory, an adaptation to the present non-compact U(1) case is rather straightforward. The result is [30]

$$\begin{aligned}\mathcal{A}(x) = {}& \frac{(-1)^n \mathrm{e}^{n+1}}{(4\pi)^n n!}\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n} F_{\mu_1\nu_1}(x) F_{\mu_2\nu_2}(x+\hat{\mu}_1+\hat{\nu}_1)\cdots\times \\ & \times F_{\mu_n\nu_n}(x+\hat{\mu}_1+\hat{\nu}_1+\cdots+\hat{\mu}_{n-1}+\hat{\nu}_{n-1}) + \partial_\mu^* k_\mu(x)\,,\end{aligned} \tag{5.12}$$

where the current $k_\mu(x)$ is a certain gauge invariant local expression of the gauge potentials. From the definition (5.10) and the gauge covariance of the Dirac operator, $D(x,y)[tA^\omega] = e^{ite\omega(x)} D(x,y)[tA] e^{-ite\omega(y)}$, we find

$$\begin{aligned}\ln\det M[A^{-\omega}] &= \ln\det M[A] - i\sum_{x\in\Gamma}\omega(x)\int_0^1 \mathrm{d}t[\mathcal{A}(x) - \partial_\mu^* k_\mu(x)] \\ &= \ln\det M[A] + i\frac{(-1)^{n+1}\mathrm{e}^{n+1}}{(4\pi)^n (n+1)!}\sum_{x\in\Gamma}\omega(x)\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n}F_{\mu_1\nu_1}(x)\times \\ &\quad \times F_{\mu_2\nu_2}(x+\hat{\mu}_1+\hat{\nu}_1)\cdots\times F_{\mu_n\nu_n}(x+\hat{\mu}_1+\hat{\nu}_1+\cdots+\hat{\mu}_{n-1}+\hat{\nu}_{n-1})\,.\end{aligned} \tag{5.13}$$

The last term is nothing but the Wess-Zumino action [32] in this U(1) gauge theory. The Wess-Zumino action takes a particularly simple form with our choice of the measure term (5.9).

With the definition of the Weyl determinant (5.10), we may carry out a study of anomalous U(1) gauge theories along the line of, say, reference [16]. In the full partition function $\mathcal{Z}$ (5.6), we can separate the integration over gauge degrees of freedom by using the Faddeev-Popov trick. Namely, we insert unity

$$\Delta[A]\int \mathrm{D}[\omega]\prod_{x\in\Gamma}\delta(\partial_\mu^* A_\mu^\omega(x)) = 1\,, \tag{5.14}$$

into $\mathcal{Z}$ where $\Delta[A]$ is the Faddeev-Popov determinant. Here, we imposed the Lorentz gauge on the lattice. In the functional integration over the gauge transformation $\mathrm{D}[\omega]$, we understand that the zero-mode of ω is excluded by the condition $\sum_{z\in\Gamma}\omega(z) = 0$. Even with this condition, there exists a "translational invariant" measure of ω such that $\mathrm{D}[\omega+\lambda] = \mathrm{D}[\omega]$. With this exclusion of the zero-mode, the Faddeev-Popov determinant becomes well-defined because for each $A_\mu(x)$ the equation $\partial_\mu^* A_\mu^\omega(x) = 0$ has a unique solution within this space of ω:

$$\omega(x) = \sum_{y\in\Gamma} G_L(x-y)\partial_\mu^* A_\mu(y)\,, \tag{5.15}$$

where $G_L(z)$ is the Green function of the lattice laplacian

$$\partial_\mu^* \partial_\mu G_L(z) = \delta_{\tilde{z},0} - L^{-2}\,, \qquad G_L(z + L\hat{\mu}) = G_L(z)\,, \qquad \sum_{z\in\Gamma} G_L(z) = 0\,. \tag{5.16}$$

As in the continuum theory, one can then confirm that the Faddeev-Popov determinant is gauge invariant $\Delta[A^\omega] = \Delta[A]$. By using the gauge invariance of $\Delta[A]$ and S_{G}, we then arrive at the expression

$$\mathcal{Z} = \int \mathrm{D}[A_\mu]\,\Delta[A_\mu] \prod_{x\in\Gamma} \delta(\partial_\mu^* A_\mu(x))\, e^{-S_{\mathrm{G}}} \int \mathrm{D}[\omega]\, e^{-S_{\mathrm{counter}}[A^{-\omega}]} \det M[A^{-\omega}]\,. \tag{5.17}$$

In what follows, we restrict our attention to 2 dimensional case ($n = 1$), i.e. the chiral Schwinger model, for which the classical continuum limit of the Weyl determinant is amenable to a simple study. Namely, it is not difficult to find that the classical continuum limit of equation (5.10) is

$$\ln\det M[A] = -\frac{\mathrm{e}^2}{8\pi}\int \mathrm{d}^2x\, A_\mu(x)\left\{\delta_{\mu\nu} - (\delta_{\mu\alpha} + i\epsilon_{\mu\alpha})\frac{\partial_\alpha\partial_\beta}{\Box}(\delta_{\beta\nu} - i\epsilon_{\beta\nu})\right\}A_\nu(x) + O(a)\,. \tag{5.18}$$

We can arrive this expression without any calculation of lattice Feynman integrals. Assuming that the rotational symmetry is restored in the continuum limit, a general argument on fermion one-loop diagrams in 2 dimension (see, for example the appendix of reference [33]) tells that regularization ambiguities can appear only in a coefficient of the local term $A_\mu^2(x)$ in equation (5.18). Then the fact that our Weyl determinant gives rise to the (consistent) gauge anomaly of the form (5.13) completely fixes the coefficient of the term $A_\mu^2(x)$ as equation (5.18). In traditional arguments in quantization of this anomalous chiral Schwinger model [15], this regularization ambiguity in the local term $A_\mu^2(x)$ is fully utilized [34]. So to incorporate this point to our formulation, we further set the lattice counter terms as

$$S_{\mathrm{counter}} = \frac{\mathrm{e}^2}{8\pi}(b-1)\sum_{x\in\Gamma} A_\mu^2(x)\,, \tag{5.19}$$

where b is a free parameter. This completes our construction of the chiral Schwinger model on the lattice. It yields

$$\begin{aligned}\mathcal{Z} = &\int \mathrm{D}[A_\mu] \prod_{x\in\Gamma} \delta(\partial_\mu^* A_\mu(x))\, e^{-S_{\mathrm{G}} - S_{\mathrm{counter}}[A]} \det M[A] \int \mathrm{D}[\omega]\,\times \\ &\times \exp\left\{-\frac{\mathrm{e}^2}{8\pi}\sum_{x\in\Gamma}[(b-1)\partial_\mu\omega(x)\partial_\mu\omega(x) - 2(b-1)\omega(x)\partial_\mu^* A_\mu(x) - i\omega(x)\epsilon_{\mu\nu}F_{\mu\nu}(x)]\right\},\end{aligned} \tag{5.20}$$

where we have got rid of the Faddeev-Popov determinant $\Delta[A]$ because it is a constant for the present Lorentz gauge. The action in equation (5.20) except the gauge fixing term

$$\begin{aligned}&S_{\mathrm{G}}[A] + S_{\mathrm{counter}}[A] - \ln\det M[A] + \\ &\quad + \frac{\mathrm{e}^2}{8\pi}\sum_{x\in\Gamma}\left[(b-1)\partial_\mu\omega(x)\partial_\mu\omega(x) - 2(b-1)\omega(x)\partial_\mu^* A_\mu(x) - i\omega(x)\epsilon_{\mu\nu}F_{\mu\nu}(x)\right]\,,\end{aligned} \tag{5.21}$$

is gauge invariant under $A_\mu(x) \to A_\mu(x) - \partial_\mu\lambda(x)$ and $\omega(x) \to \omega(x) + \lambda(x)$ because it was originally $S_{\mathrm{G}}[A] + S_{\mathrm{counter}}[A^{-\omega}] - \ln\det M[A^{-\omega}]$ and the combination $A^{-\omega}$ is invariant under the transformation. After integrating over $\omega(x)$, we have

$$
\begin{aligned}
\mathcal{Z} = \int \mathrm{D}[A_\mu] \prod_{x\in\Gamma} \delta(\partial_\mu^* A_\mu(x)) \times \\
\times \exp\left\{ -\frac{1}{4g_0^2} \sum_{x\in\Gamma}\sum_{\mu,\nu} \mathcal{L}_{\mu\nu} + \frac{m^2}{4g_0^2} \sum_{x,y\in\Gamma} F_{\mu\nu}(x) G_L(x-y) F_{\mu\nu}(y) \right\} \det M[A] \times \\
\times \exp\left\{ \frac{\mathrm{e}^2}{8\pi} \sum_{x,y\in\Gamma} A_\mu(x) \left[\delta_{\mu\nu}\delta_{x,y} - (\partial_\mu^x + i\epsilon_{\mu\alpha}\partial_\alpha^{x*}) G_L(x-y) (\partial_\nu^{y*} - i\partial_\beta^y \epsilon_{\beta\nu}) \right] A_\nu(y) \right\} \times \\
\times \exp\left\{ -\frac{\mathrm{e}^2 b}{8\pi L^2} \left[\sum_{x\in\Gamma} A_\mu(x) \right]^2 \right\}, \qquad (5.22)
\end{aligned}
$$

where the mass parameter m^2 is given by $m^2 = (\mathrm{e}^2 g_0^2/4\pi) b^2/(b-1)$. In an intermediate stage in deriving the above expression, we have used the identity $\partial_\mu\partial_\nu^* = \delta_{\mu\nu}\partial_\rho^*\partial_\rho + \epsilon_{\mu\rho}\partial_\rho^*\partial_\sigma\epsilon_{\sigma\nu}$ holding on a 2 dimensional lattice. It is easy to verify that the action in the above expression is still gauge invariant. From equations (5.8) and (5.18), the classical continuum limit of the above action is given by

$$
-\frac{1}{4g_0^2} \int \mathrm{d}^2x\, F_{\mu\nu}(x) \frac{\Box - m^2}{\Box} F_{\mu\nu}(x) = -\frac{1}{2g_0^2} \int \mathrm{d}^2x\, A_\mu(x)(-\Box + m^2) A_\mu(x)\,, \qquad (5.23)
$$

under the gauge condition $\partial_\mu A_\mu(x) = 0$. This appearance of massive excitations in the anomalous chiral Schwinger model is in accord with the result of references [15, 16].

6. Discussion

In this paper, we showed that a consistent lattice formulation of anomalous chiral gauge theories along the line of framework of references [1, 2] is rather difficult and for a certain class of models it is in fact impossible. For an implication of our observation, one can take two alternative attitudes. One may of course regard obstructions for a smooth fermion measure we discussed as a pathology being peculiar to the present lattice framework based on the Ginsparg-Wilson relation. For example, it might be possible to formulate anomalous non-abelian chiral theories by using certain non-compact gauge variables, as we have demonstrated for the abelian case. With such non-compact gauge variables, however, an issue of how a non-trivial topology of gauge field configurations emerges on a lattice has to be clarified.

Alternatively, it is possible to take our observation rather seriously and consider its possible implications. We may regard it as an indication for an impossibility of a consistent quantization of anomalous chiral gauge theories in general. (The successful treatment of abelian theories in section 5 is regarded exceptional.) We emphasize that obstructions we observed do not exist in effective low-energy theories with an anomalous fermion content, which are obtained by sending mass of some of fermions very large (say, by sending the expectation value of the Higgs field very large) from an underlying anomaly-free theory in which the gauge anomaly is cancelled among flavors of the Weyl fermion. From the way of construction in section 2, the effect of the measure term is left over even if the mass of a Weyl fermion is sent to infinity (like the Wess-Zumino term in the continuum theory [19]).[14] Then the obstruction we discussed is cancelled among fermion flavors provided that the gauge anomaly is cancelled among them. The effect of the measure term does not decouple even the mass of fermion is very large.

This latter attitude is compatible with a general belief that anomalous gauge theories are inconsistent anyway and the fundamental theory must be free from anomalies after all. This is OK. However, a trouble with the latter attitude is the fact that a cancellation among flavors of the Weyl fermion is *not* an only known way for a cancellation of the gauge anomaly. The Green-Schwarz mechanism [35] is another known way for the anomaly cancellation. In this mechanism, the anomaly arising from Weyl fermions is cancelled by bosonic anti-symmetric tensor fields by assuming a non-trivial gauge transformation law of latter fields. Then the question is: can we set up a non-perturbative lattice formulation of anomaly-free theories based on the Green-Schwarz mechanism? In the context of the present lattice framework based on the Ginsparg-Wilson relation, to answer this question in a positive way, we have to find a certain interplay between the measure term of Weyl fermions and a definition of anti-symmetric tensor fields on a lattice. Such an interplay seems not straightforward to find at the moment. We have to admit, therefore, the situation is not clear for chiral gauge theories with an anomalous fermion content in which the anomaly *is* cancelled by the Green-Schwarz mechanism.

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Note added in proofs. The partition function of Weyl fermions in an anomalous multiplet can always be made gauge invariant by introducing a dynamical Wess-Zumino scalar. This way of anomaly cancellation can be regarded as the simplest form of the Green-Schwarz mechanism. At least this form of the Green-Schwarz mechanism can readily be implemented in lattice gauge theories by (1) replacing link variables $U(x,\mu)$ by $U^g(x,\mu)=g(x)U(x,\mu)g(x+\hat\mu)^{-1}$ and (2) integrating over the Wess-Zumino scalar g with an appropriate gauge invariant kinetic term. Under the gauge transformation, $\delta U(x,\mu)=-\nabla_\mu\omega(x)U(x,\mu)$ and $\delta g(x)=-g(x)\omega(x)$, the combination U^g is gauge invariant and thus the partition function of the fermion sector is trivially gauge invariant. We then examine a smoothness of the fermion measure as a function of configurations of the gauge field and of the Wess-Zumino scalar. The curvature of the U(1) bundle associated to the fermion measure defined over this "enlarged" configuration space is given by the right hand side

[14] The Yukawa interaction can be introduced without affecting to the chirality constraint (2.5).

of eq. (2.13) with substitutions $U \to U^g$, $\eta_\mu \to g\eta_\mu g^{-1}$ and $\zeta_\mu \to g\zeta_\mu g^{-1}$. Because of the gauge covariance of the projection operator $\hat{P}_-$, however, this curvature turns out to have an identical form as in the case without the Wess-Zumino scalar. Hence, the obstructions for a smooth fermion measure we discussed in this paper remain in this enlarged theory, although the gauge anomaly is exactly cancelled. We conclude that a large class of anomalous chiral gauge theories, even with this simplest form of the Green-Schwarz mechanism, cannot consistently be formulated in the present lattice framework.

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